TI-nspire™

The Witch of Agnesi

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Grade level: secondary (Years 9-12)
Subject: mathematics
Time required: 90 minutes

Overview

This activity involves a curve "with history"; an excellent example of combining geometry and algebra. This function, the so-called "Witch of Agnesi" is defined by a geometric description. After implementing the construction, students are then challenged to find the equation of the constructed curve. This equation, in turn, allows further investigations and generalisations, including some from the field of analysis - an alternative to conventional curve sketching.

The first and second parts of the task are suitable for students of secondary school age with knowledge of the theorems of intersecting lines and the laws of similarity, as well as the Pythagorean theorem. Methods of differential calculus are required only for the final task.

Maria Gaetana Agnesi
(1718-1799)

Background

In 1750, Maria Gaetana Agnesi was the first woman to be appointed a professor of mathematics at the University of Bologna. She investigated this curve in her book *"Instruzioni Analitiche"* of 1748. In Old Italian, the original name *Versiera* means both "flexible" and "witch".

Mathematical content

Combination of geometry and algebra, theorems on the intersection of lines (Triangle Propotionality Theorem[1]), circle equation (Pythagorean theorem), drawing of function graphs, function investigations with and without differential calculus.

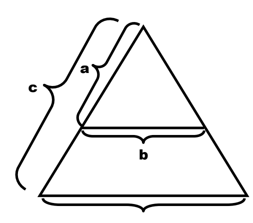
Classroom Management Tips

The geometric construction and derivation of the function should be carried out at all levels. The generalisation for a family of curves, and the corresponding function investigations in Part 3 are more suitable for teaching at senior levels.

It helps if students have previous experience with interactive geometry software. It is also a good idea for students to work initially with paper and pencil, to get an overview of the construction before implementing it in Graphs & Geometry. The construction can also be carried out with a circle of any radius. However it saves time to agree on a specific radius (e.g. r = 1).

In this phase of the lesson, students should spend approximately 10 minutes examining the construction by themselves. Work should subsequently be carried out in groups of two to four to minimise failures. As teacher you should also make sure that every student actually carries out the construction on their computer or handheld. A student demonstration ensures the quality of interim results.

Group-work is also suitable for the subsequent investigations. If necessary the teacher can provide information, but should give students the space to find their own solution paths. They can present their results, or interim results if applicable, in front of the study group.



$$\frac{a}{c} = \frac{b}{d}$$

Ensure that solution paths are adequately documented, either in Notepad or in written form.

Technical requirements

Students should know how to:

- *carry out simple geometric constructions with interactive geometry software(points, lines, circles, loci, etc.) and implement these in **Graphs & Geometry**,*
- *draw a graph,*
- *solve equations in the **Calculator** application and define function equations,*
- *transfer point coordinates in a table and draw a scatter plot,*
- *precisely and numerically define surface area (**Calculator and Graphs** & **Geometry**).*

Step-by-step instructions

Part 1: Construction of locus curve

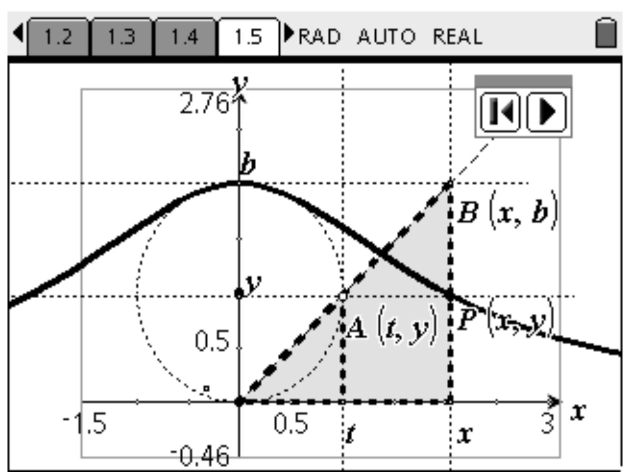
1. Briefly present the task and explain the work assignments:
   (i) Familiarise yourself with the construction text (a pencil sketch helps).
   (ii) Then construct the curve with the software or handheld. Select a circle with the radius r = 1.

   First let the students work individually and, if possible, give them no information as to the appearance of the construction.

   Possible errors: Students take Point *A* as the point of intersection between circle and line. In this case *A* can no longer be moved. *A* must first be placed on the circle and then the line through the origin drawn.

2. Encourage students to demonstrate their construction.

Construction instructions*:*
Draw a circle which meets the horizontal axis at the origin, as well as a parallel to this axis through the highest point of the circle.
A line through the origin intersects the circle in A(t,y) and the parallel in B(x,b). A point P(x,y) is determined by A and B. This point P defines a curve, i.e. if A moves on the circle, the locus of P is the required curve.

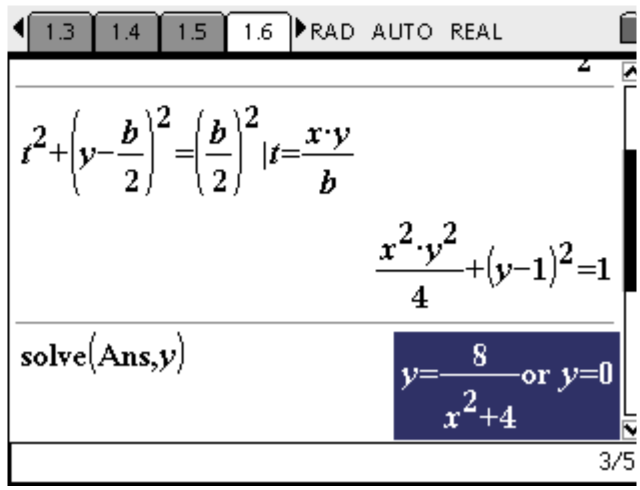


Part 2: Formulation of the corresponding equation

1. Next, ask students to derive an equation in the form of y = f(x) from the geometric data. For this, work should be carried out in small groups (2 - 4 group members). This also applies to all other work assignments. A precise legend for the design drawing using the identifiers x, y, t and b is helpful.

2. Students can test their solution by changing the attributes of the locus curve. Does *P* move on the graph if we drag *A*? Make sure that the units in x- and y-direction have the same scales.

3. Students present and explain their results.



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Part 3: Function investigations

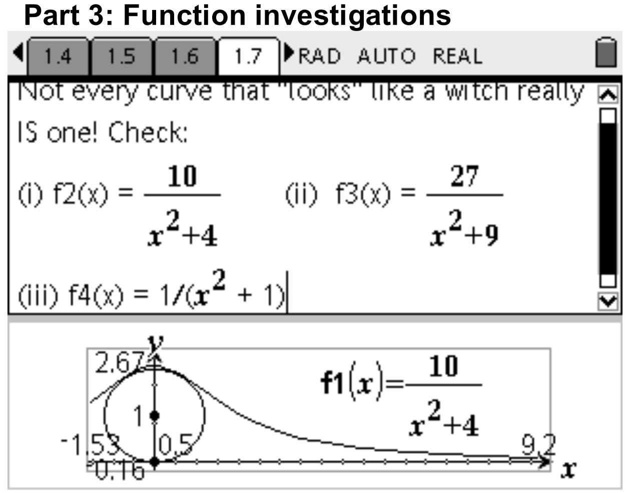
1. Ask students for further investigations.

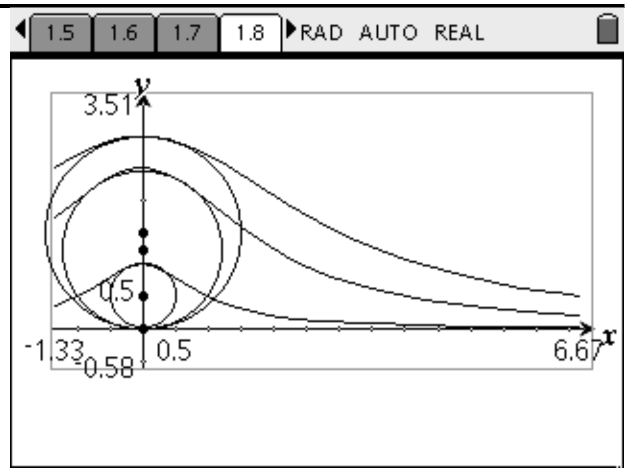
   Question: Which of the following functions $f_2(x)=\frac{10}{x^2+4}$, $f_3(x)=\frac{27}{x^2+9}$, $f_4(x)=\frac{1}{x^2+1}$ involves a "witch"?

   In this case various solution paths are possible. For example, students draw the graph and try to construct an appropriate circle. This does not work for $f_2(x)$.

   The functions f_3 and f_4 are both "witches", f_3 has a circle where r =1.5; f_4 has a circle where r = 0.5. Here we do not prove that the curve and the circle have only one common point.

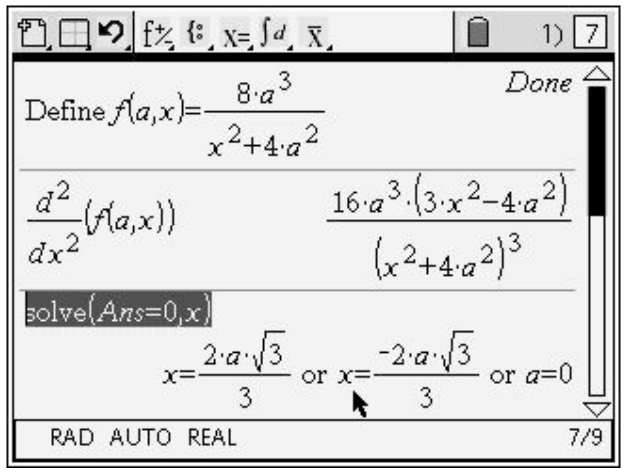
   A further opportunity for investigation could be to establish a general function for a circle with radius *a*. This leads to the question examined in the next section.





2. This provokes the following question: which equation belongs to a circle with radius *a*?

   The derivation can also be deduced using the Triangle Proportionality Theorem. This generalisation is the basis for a family of functions.



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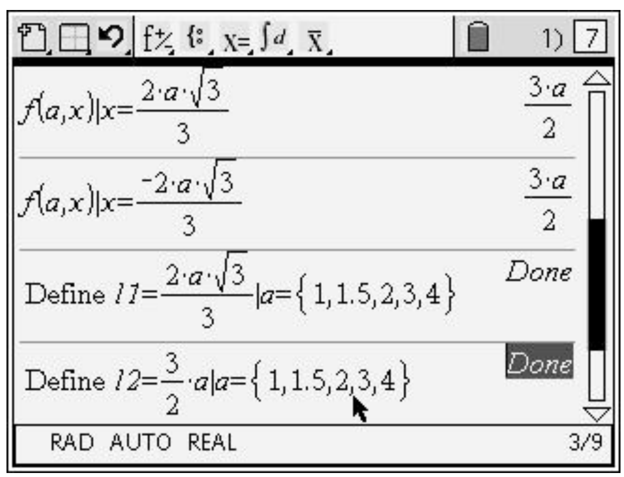
3. The position of the inflexion point is important in conducting a more precise investigation. These inflexion points can be determined using the **Calculator** application.

   To find the curve on which all inflexion points lie, students might draw in some points as a scatter plot on the graph.

   Press the left function input icon in **Graphs & Geometry** several times until scatter plot appears. Enter the list names for *x* and *y*.



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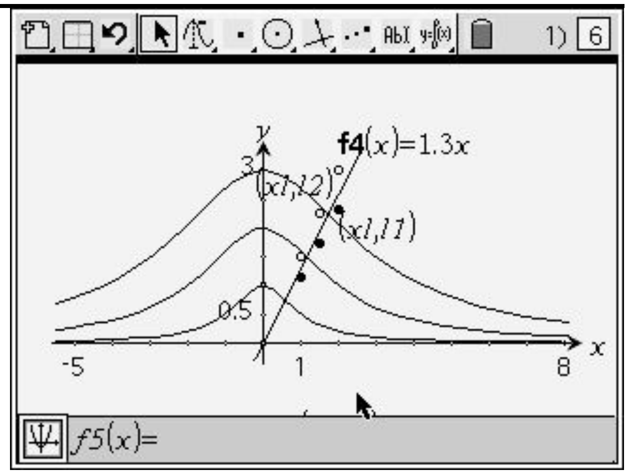
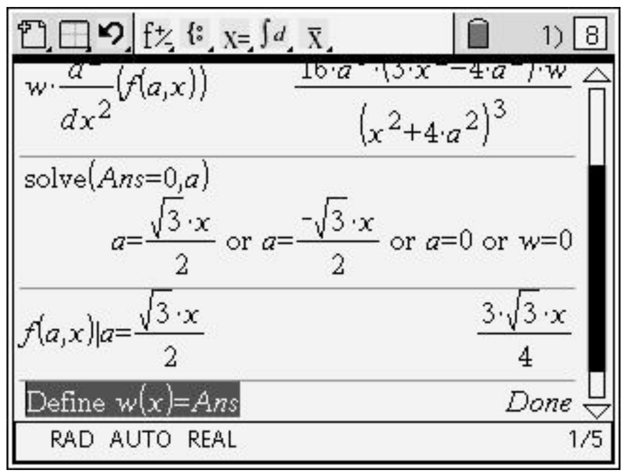
4. One possible conjecture is that the inflexion points lie on two lines which pass through origin.
   To obtain the equation for these lines, students could work with the regression module or demonstrate the equations of the lines drawn.
   Discuss the various solution paths without devaluing empirical numerical solutions.

The "classical" path is an alternative route to take: Parameter *a* is eliminated due to determination of the inflexion point from the function equation.





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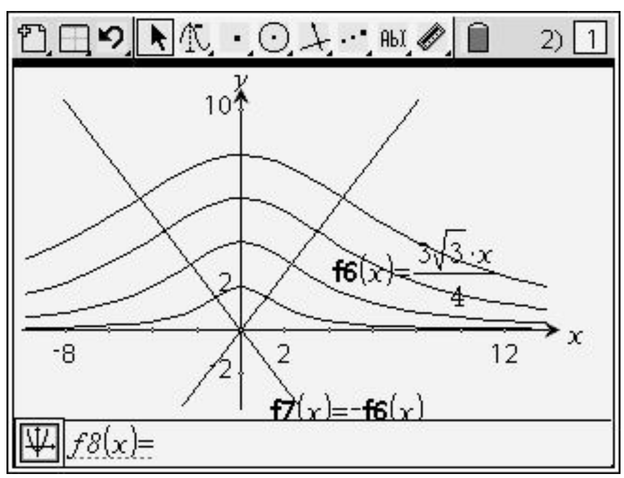


Assessment and evaluation

The task must be differently weighted according to the age of the study group. In the middle secondary years, the emphasis lies on the construction and the derivation of the function equation. Questions 1 and 2 in the section 'Function investigations' are suitable if derivatives are available.
Students at senior level should also carry out the construction themselves. The time requirement is difficult to assess, but the compulsory section should include questions 1 and 2 on function investigations. In every case, the teacher should discuss the link between geometry and algebra.

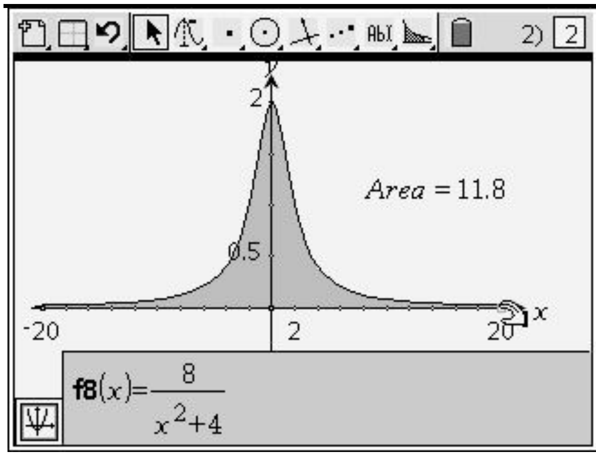
For almost all questions there is more than one solution path. To clarify the process orientation of the mathematics lesson, these should also be presented and compared.
Student presentations backing up interim results from which a conclusion is drawn, enhance understanding of interrelations and give the opportunity for subsequent questions and discussions.

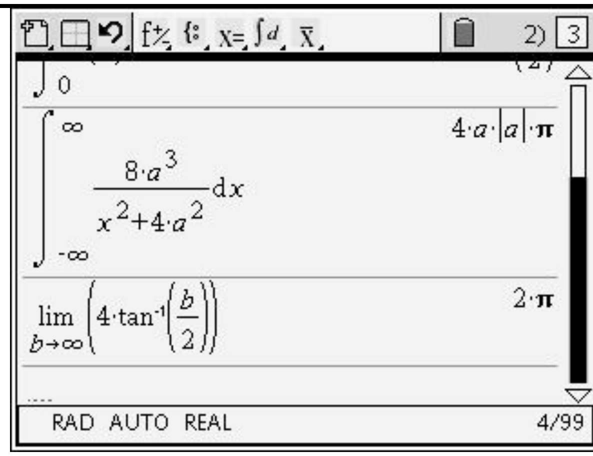
The **solutions** to the individual tasks can mainly be found in the accompanying screenshots.

1. Carry out the specified construction.	1. Construct the circle and the line parallel to the x-axis. Place *A* on the circle and draw a line through (0,0) and A. You intersect the parallel in *B*. Draw the corresponding parallels to the axes. Draw the locus of the point of intersection *P* by moving *A*.
2. Find an equation for the constructed curve.	2. The equation is derived using a combination of the Triangle Proportionality Theorem and the circle equation: $f(x)=\frac{8}{x^2+4}$
3. Derive the equation for a circle with radius r = a.	3. $f(a,x)=\frac{8\cdot a^3}{x^2+4\cdot a^2}$
4. Calculate the inflexion points of the family of curves.	4. The following applies to the inflexion points: $W=(\pm\frac{2a\sqrt{3}}{3}\mid\frac{3a}{2})$.
5. On which curve does the inflexion point lie?	5. The inflexion points lie on the lines $w_1(x)=\frac{3\sqrt{3}}{4}x$ and $w_2(x)=-\frac{3\sqrt{3}}{4}x$.

For further consideration

Calculate the area between the curve and the x-axis for *a* = 1 and then for any value of *a*. Where does the figure π come from in the integration result?





Applications of the Witch

It has been noted that, for a curve that has generated so much interest over several hundred years, it is very difficult to find any real-world applications. In fact, while this curve does prove elusive in its links to the world around us, there are several applications which may be found if students exercise a little diligence in their searching. *Google* searches will readily reveal links to physics and optics in the form of spectral radiation, but other applications may be found to such important topics as **resonance, fluid dynamics** and **probability**! In fact, our Witch goes under several names, some of which are well known. These include the Cauchy Distribution and perhaps even the Student's t-distribution for 1 degree of freedom! Of course, we do well to remember that not every curve the looks like a witch actually is a witch – so perhaps some further investigation would be worthwhile here to verify these relationships!

Some References

Applications of the Witch of Agnesi:
http://www.mathsci.appstate.edu/~sjg/wmm/final/agnesifinal/applications.pdf

Resonance Curve and Cauchy Distribution: *http://www.2dcurves.com/cubic/cubicr.html*

Cauchy, Agnesi and Student's-t: *http://www.pballew.net/arithme5.html*

